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NONLINEAR ESTIMATION APPLIED TO THE NONLINEAR INVERSE HEAT CONDUCTION PROBLEM

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Abstract—The estimation of the surface temperature or heat flux density utilizing a measured temperature history inside a heat-conducting solid is called the inverse heat conduction problem. This problem becomes nonlinear if the thermal properties are temperature-dependent. A new finite-difference method is given. It is based in part upon the concepts of a general technique for solving inverse problems called nonlinear estimation.

The method (or family of methods) estimates the components of the heat flux one at a time and thus, may be considered an on-line method. Another method is outlined for which all of the components of the heat flux are found simultaneously.

As suggested by the developments in nonlinear estimation, the sensitivity coefficients can be utilized to gain insight into these methods. The sensitivity coefficients help indicate that the on-line method requires "future" temperatures when small time steps are to be used.

Several examples of the use of the on-line method are given. These examples are for cases with non-exact data. The results demonstrate a method that is rather remarkable in its ability to extract information about the surface condition from experimental measurements that lag and are damped compared to the surface condition.

NOMENCLATURE

- c_p, specific heat;
- E, distance from heated surface to temperature sensor;
- F, sum of squares function, equation (3);
- I, upper limit for F, equation (6);
- k, thermal conductivity;
- L, thickness of solid;
- m, integer, see equation (5a);
- M, subscript for q indicating the last known q;
- P, coefficient in energy equation, (1);
- q, heat flux density at $x = 0$;
- Q, volume heat source;
- r, small integer; if $m = 1$, $r - 1$ = no. of "future" temperatures;
- t, time;
- T, temperature;
- x, coordinate;
- Y, experimental temperature;
- α, thermal diffusivity;
- η, time index for t and T;
- θ, time scale for q;
- ρ, density;
- τ, dimensionless time;
- ϕ, sensitivity coefficient, equation (8b).

INTRODUCTION

In evaluating new heat-shield materials, testing of rocket nozzles and developing transient calorimeters, it is sometimes necessary to calculate the transient surface heat flux and the surface temperature from a temperature history measured at some location inside the body. In distinction to usual transient heat conduction or diffusion problems this one has been termed the inverse problem. This is because the conditions are not specified at both boundaries. If the thermal properties are functions of temperatures as considered in this paper, the inverse problem becomes nonlinear.

One of the first papers on this subject was written by Stolz [1]. His procedure is unstable if the time intervals are made too small. Small time steps in the calculations are needed to obtain more information about the surface conditions. Much smaller time steps were found by Beck to be possible by utilizing least squares [2]. Other papers using least squares in quite different ways were written by Frank [3] and Burggraf [4]. Sparrow *et al.* [5] used another approach. Using different approximations each of these researchers except Burggraf and Frank found the average heat flux for a succession of short intervals. Burggraf cleverly found the exact solution for the instantaneous surface heat flux for given *continuous* temperature and heat flux histories at a given internal point. When Burggraf's equation is utilized with discrete or experimental data, the results are also approximate.

Frank and Davies [6] assume an analytical form for the time-variations of the surface heat flux for the duration of the experiment. These investigators do not report any problems with stability, but their methods are not designed for determining heat flux curves which may be discontinuous.

None of these papers presents a method which can conveniently heat a composite body with temperature-variable thermal conductivity and specific heat. An objective of this paper is to present a method suitable for use with a digital computer for treating such a body. In [7] a method for the same problem is discussed which builds on the ideas in [2]. In both [2] and [7] the methods are not developed using the ideas of nonlinear estimation to which they are in fact related. Another objective then is to derive the methods using nonlinear estimation in order to demonstrate the applicability of nonlinear estimation to determining time-dependent quantities at a boundary. By incorporating this problem into the framework of nonlinear estimation, the analytical solution of this problem can be improved. Furthermore, the similar problems of determining the heat-transfer coefficient, thermal contact conductance, mass-transfer coefficient, etc., can be treated in an almost identical manner.

Nonlinear estimation is not well-known by mechanical engineers. It is presently being actively developed by men of diverse backgrounds who seem sometimes unaware of the work of each other. A great deal has been contributed by control and chemical engineers, mathematicians and statisticians. Related to nonlinear estimation are terms such as nonlinear least squares, optimization, identification, filtering, quasilinearization and invariant imbedding.

VARIOUS METHODS OF SOLUTION

The linear inverse problem in conduction can be solved using an exact method [4], integral method [1, 2, 5] or a finite-difference method [3, 6, 7]. The only method sufficiently powerful to solve the general nonlinear problem appears to be the latter; hence, the discussion will emphasize this method although the basic concepts can be applied to the integral method.

One can use (a) a step-by-step method to calculate the heat flux q-distribution one component at a time [1, 2, 4, 5, 7, 8] or (b) obtain the complete q-distribution simultaneously at the end of the procedure [3, 6]. The relative merits of each is discussed further in later sections.

For any method given above except in [4] an analytical form for the time-variation of q must be chosen. This choice is particularly critical for case (b). Frank [3] suggested a polynomial approximation of kth degree in time for q. Polynomials have the advantage that the computational procedure is relatively simple. Severe disadvantages of polynomials include their inability "to take on sharp bends followed by relatively flat behavior" and, as the degree of a polynomial increases, they are frequently "difficult to evaluate, i.e. being numerically unstable" [9]. Flat bends followed by flat behavior are relatively common for such situations typified by starting and stopping a plasma arc heat source. Davies [6] uses rather a complicated form for q involving error functions and nine unknown coefficients. His form is difficult to solve and may well have some of the disadvantages of Frank's method.

The determination of the heat flux simultaneously at the completion of a calculation, case (b), need not be restricted to functions which are continuous over the duration of the experiment. The q-curve could be approximated by an array of values of q, each corresponding to different times, for example.

PROBLEM DESCRIPTION

The heat-conduction equation can be generalized as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \rho c_p \frac{\partial T}{\partial t} - P\frac{\partial T}{\partial x} - Q. \qquad (1)$$

The last two terms on the right-hand side of equation (1) relate respectively to a fluid flowing through the solid and to the rate of production of internal energy per unit volume. If a gas is traveling from the rear surface ($x = L$, Fig. 1)

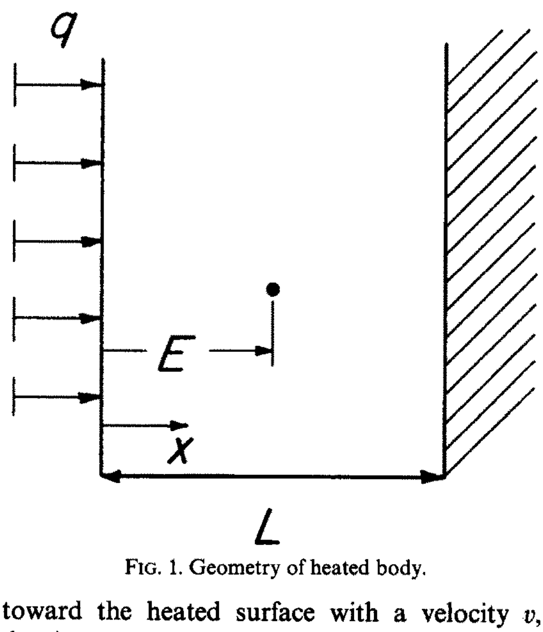

FIG. 1. Geometry of heated body.

toward the heated surface with a velocity v, density ρ_g and specific heat c_{pg}, then P is equal to $\rho_g c_{pg} v$. The quantities k, c_p, P and Q can be temperature- and space-variable.

The problem is to calculate the heat flux $q(0, t)$ given the temperature history at $x = E$, Fig. 1. For convenience assume the boundary at $x = L$ is insulated; any other known boundary condition could be used, however. Mathematically the boundary and initial conditions are

$$T(E, t) = Y(t), \qquad (2a)$$

$$\frac{\partial T(L, t)}{\partial x} = 0, \qquad (2b)$$

$$T(x, 0) = T_i(x). \qquad (2c)$$

The temperatures $Y(t)$ and $T_i(x)$ are known. The unknown surface heat flux $q(0, t)$ and surface temperature $T(0, t)$ are simultaneously found while calculating the temperature distribution in the body.

The numerical solution of equations (1) and (2) can be accomplished by using a number of finite-difference approximations. A backward-difference formulation is given in [7]. See also Ames [10]. The problem from $x = E$ to $x = L$ can be solved in a straightforward manner because the boundary conditions are known at each boundary. The solution of the complete problem from $x = 0$ to L cannot be readily obtained because the boundary condition is not given at $x = 0$ but rather an interior temperature history is given; because the boundary condition is to be determined, the problem is the inverse of the usual one.

Difficulties arise in determining the surface heat flux or temperature from the physics of the problem. That is, the temperature response at any interior location is damped and delayed (in effect) with respect to the heated surface [7]. As one seeks to determine the structure of $q(0, t)$ more finely (i.e. using smaller time increments), the effect of the physics is to make the calculation more sensitive to errors. The exact solution of Burggraf for the linear case can be used to show that as the time steps are made smaller, higher and higher orders of time derivatives of the measured temperature $Y(t)$ and of the heat flux at $x = E$ become dominant. Note that these derivatives must be taken for discrete experimentally measured data; this can only be approximated by replacing the derivatives by differences.

A limit on the minimum size of the time increments is sometimes caused by the stability of the numerical procedure. Using some procedures, for example [1], this minimum permissible time step may be too large to extract all the useful information out of the data. Stable procedures can be devised, however, to permit passing to even smaller time steps. Since the information contained in the data is finite, it is not necessary to present a procedure which is stable as the time steps go to zero.

It can be readily verified from the Burggraf solution or the discussion in [7] that the heat flux at the instant t is dependent upon temperatures at $x = E$ at times greater than t. Hence, in determining $q(0, t)$, any effective procedure would utilize temperatures at times greater than t. Nonlinear estimation permits one to do this in an effective manner.

NONLINEAR ESTIMATION PROCEDURE FOR SINGLE q

In the nonlinear estimation procedure one minimizes the sum of squares function

$$F(\bar{q}) = \sum_{i=1}^{I} (T_{\eta+i} - Y_{\eta+i})^2 \quad (3)$$

with respect to some parameters describing $q(0, t)$. Rather than approximating $q(0, t)$ as a function continuous in time such as a power series, q will be represented by a vector of elements $\bar{q}$, $(q_1, q_2, \ldots, q_N)$ which is more flexible and powerful. The simplest way to approximate $q(0, t)$ with these q_n is to let each one represent a step; that is

$$q(0, t) = q_n \quad \text{for} \quad \theta_{n-1} < t < \theta_n. \quad (4)$$

Figure 2 shows a typical $q(0, t)$ approximated with some q_n. The $Y_{\eta+i}$ values are measured temperatures at $(E, t_{\eta+i})$. The temperature $T_{\eta+i}$ is calculated using a finite difference solution for equation (1) with appropriate boundary and initial conditions. (The method of

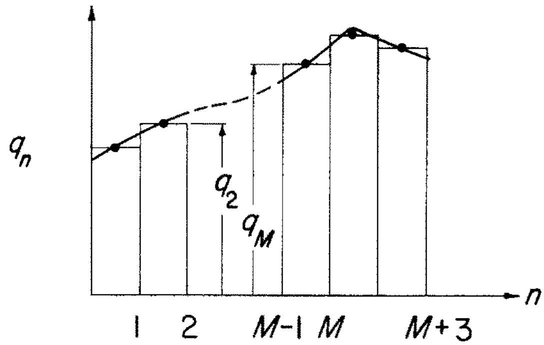

FIG. 2. Approximation of heat flux by discrete values of q.

determining the boundary condition at $x = 0$ is discussed below.) $T_{\eta+i}$ is also for time $t_{\eta+i}$ and position $x = E$.

The subscripts on q and T (or Y) do not have the same meaning although both refer to time. A reason for this is that one might decide to calculate fewer values for q (i.e. N) than the number of temperature measurements. That is, the time steps for T, Y and q are respectively Δt, Δt and $\Delta\theta$ which are related by

$$\Delta\theta = m\Delta t, \quad m \text{ is an integer.} \quad (5a)$$

Thus, θ_M and t_η are referring to the same time if

$$mM = \eta. \quad (5b)$$

(Actually a time step smaller than Δt might be used for improving the accuracy in calculating T, but only those T-values occurring at times at which Y-values are available would be introduced into the sum of squares.) For the special case of $m = 1$—same number of q_n's as T_η's—(5b) gives $\eta = M$. Figure 3 shows $T_{\eta+i}$ as a function of $\eta + i$; also shown in an n axis for $m = 2$.

The following discussion in this section refers to determining one q_n at a time. Assume that $q_1, q_2, \ldots, q_M$ are known. In (3) let

$$I = mr \quad (6)$$

where r is some small integer like 1, 2, 3 or 4. The objective is to calculate q_{M+1} using temperatures $T_{\eta+1}, T_{\eta+2}, \ldots, T_{\eta+m}$ (which are produced during the time interval associated with

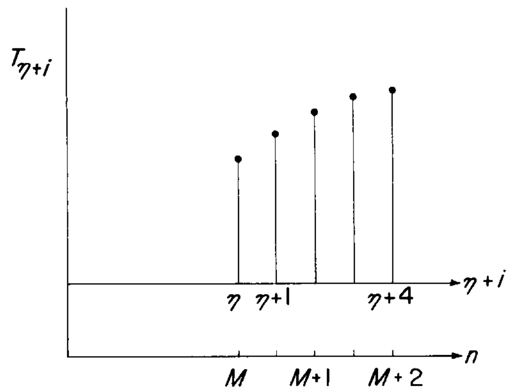

FIG. 3. Temperatures $T_{\eta+i}$ shown as a function of $t_{\eta+i}$ and θ_η for $m = 2$.

q_{M+1}) plus the temperatures $T_{\eta+m+1}, \ldots, T_{\eta+I}$. The latter temperatures might be called "future" temperatures.

For temperature-variable properties, the problem of determining q_{M+1} is nonlinear, but it can be solved by using iteration with a linear approximation. A key assumption used temporarily is to let

$$q_{M+2} = q_{M+3} = \ldots = q_{M+r} = q_{M+1} \qquad (7)$$

which sets some future q's equal to q_{M+1}. (This assumption is examined below.) Then for the lth iteration the Taylor series approximation

$$T^l_{\eta+i} \approx T^{l-1}_{\eta+i} + \frac{\partial T^{l-1}_{\eta+i}}{\partial q^l_{M+1}}(q^l_{M+1} - q^{l-1}_{M+1}) \qquad (8a)$$

is used. This expression is exact (hence, iteration is not needed) if the temperature problem is linear. The l superscript is an index related to the number of iterations. For $l = 0$, an estimate of q^0_{M+1} is the converged value of q_M. For q^0_1 one can use the value of unity.

The partial derivative in (8a) is frequently called a sensitivity coefficient. It can be calculated using

$$\phi^{l-1}_i \equiv \frac{\partial T^{l-1}_{\eta+i}}{\partial q^l_{M+1}} \approx \frac{T_{\eta+i}(q^{l-1}_{M+1}(1+\varepsilon)) - T_{\eta+i}(q^{l-1}_{M-1})}{\varepsilon q^{l-1}_{M+1}} \qquad (8b)$$

where ε is some small number such as 0·001. For convenience, the notation for the sensitivity coefficient ϕ^{l-1}_i given in (8b) does not contain either M or η; it is true, however, that ϕ^{l-1}_i is independent of M (or η) and l for the linear problem.

Introducing (8a) in (3), differentiating with respect to q^l_{M+1} gives for the correction in q^l_{M+1}

$$\nabla q^l_{M+1} = \frac{\sum_{i=1}^{I} (Y_{\eta+i} - T^{l-1}_{\eta+i})\phi^{l-1}_i}{\sum_{i=1}^{I} (\phi^{l-1}_i)^2} \qquad (9a)$$

where

$$\nabla q^l_{M+1} = q^l_{M+1} - q^{l-1}_{M+1}. \qquad (9b)$$

If the thermal properties are temperature-independent (linear problem), the maximum value of l is unity; for a nonlinear problem frequently $l = 1$ or 2 would be satisfactory. More generally, the iteration on l using (9a) could continue until, say

$$\frac{\nabla q^l_{M+1}}{q^{l-1}_{M+1}} < 0{\cdot}005. \qquad (9c)$$

The iterative use of (9a) result in the calculation of q_{M+1} which then provides a conventional boundary condition at $x = 0$ for $\theta_M < t < \theta_{M+1}$

Several special cases can be obtained from (9a). One such case is for $m = 1$; thus,

$$\Delta\theta = \Delta t \qquad (9d)$$

and the subscripts of q and T when equal refer to the same instant. Then (9a) becomes

$$\nabla q^l_{M+1} = \frac{\sum_{i=1}^{r} (Y_{M+i} - T^{l-1}_{M+i})\phi^{l-1}_i}{\sum_{i=1}^{r} (\phi^{l-1}_i)^2}. \qquad (10)$$

This result is analogous to equation (9a) in [7]. For $r = 1$ (no future temperatures used), least squares would not have been needed. For $r > 1$, future temperatures are used.

Another special case obtained from (9a) is for $r = 1$ and thus, $I = m$. In this case no "future" temperatures are used but the above procedure is needed to obtain the result if $m > 1$. The expression obtained in this case is analogous to equation (6) of [11] which gives a method for determining the heat-transfer coefficient or contact conductance from transient temperatures in a solid. The time interval associated with q_n which is $\Delta\theta$ should be larger when no future temperatures are used than when $r = 2$ or 3.

If several thermocouples are used to calculate the surface heat flux, then exactly the same procedure may be followed. The only difference would be that the summation in (9a) would be replaced by a double summation over time and thermocouples.

A careful investigation of the sensitivity coefficients often can aid in providing insight into the problem of determining parameters using nonlinear estimation. For this reason ϕ_i will be examined. One of the desired characteristics of a sensitivity coefficient in parameter estimation problems is that the magnitude of the coefficient be as large as possible. If there are two or more parameters to be found at one time, then in addition the sensitivity coefficients should be as uncorrelated as possible [1].

For convenience in this analysis relating to the sensitivity coefficients, the properties will be assumed to be constant. Using the method given in [11] it can be shown that

$$\phi_i^* \equiv q_{M+1} \frac{\partial T_{\eta+i}}{\partial q_{M+1}} = G(x, t_{\eta+i} - t_\eta) \qquad (11)$$

where $G(x, t)$ is the temperature rise in a body due to a unit step change in $q(0, t)$ at time t_η. Equation (11) applies quite generally; it is valid for any one-dimensional problem (with P and $Q = 0$ and temperature-independent k and c) including finite and semi-infinite bodies, radial and spherical geometries, and composite bodies arranged to permit one-dimensional heat flow.

To utilize (11) consider a homogeneous, plane one-dimensional body heated at $x = 0$ and insulated at $x = L$. The solution for a step change in q is well known [12] and is shown in Fig. 4. Observe the character of the curve for $x = 0$; the sensitivity coefficient immediately begins to rise at $t_i - t_\eta = 0$. Hence, thermocouples placed at the surface are immediately responsive to changes in the surface heat flux.

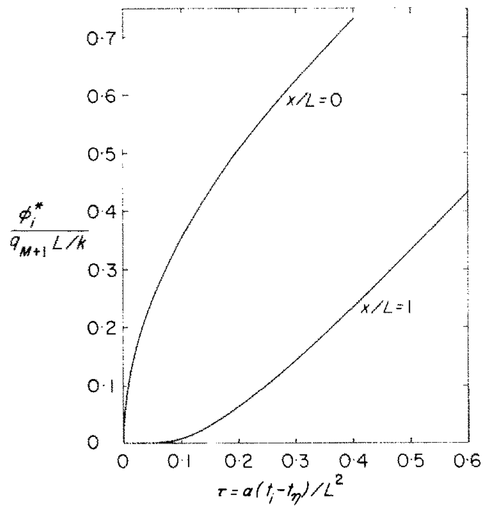

FIG. 4. Dimensionless sensitivity coefficients for step change in q as a function of time.

On the other hand, for any interior location there is a lag in the response of ϕ_i^*. It is clear that if the sensitivity coefficients are very small, as for $x/L = 1$ for

$$\tau = \frac{\alpha(t_i - t_\eta)}{L^2} < 0{\cdot}06 \qquad (12)$$

that from (8a) $T_{\eta+i}$ is relatively independent of q_{M+1} and further that difficulties are going to arise using (9a). These difficulties probably will manifest themselves as oscillations in the calculation.

To be more specific, (9a) for this case can be written for $m = 1$

$$\frac{\nabla q_{M+1}}{q_{M+1}} = \frac{\sum_{i=1}^{r} (Y_{M+i} - T_{M+i})\, \phi_i^*}{\sum_{i=1}^{r} (\phi_i^*)^2}. \qquad (13)$$

Let $\Delta\tau = 0{\cdot}05$ be the time interval for q_{M+1} and let $m = 1$. Then $\phi_1^* = 0{\cdot}0002$, $\phi_2^* = 0{\cdot}0078$ and $\phi_3^* = 0{\cdot}0294$ for $qL/k = 1$ F° and $x/L = 1$. Since ϕ_1^* is much smaller than ϕ_2^* or ϕ_3^*, much more information relative to q_{M+1} (which is applied, in terms of Fig. 4, from $\tau = 0$ to $\Delta\tau$) is obtained after $\tau = \Delta\tau$ than before this time. In Fig. 5 shown by the dots are some values of ϕ_1^* for $\Delta\tau = 0{\cdot}05$, 0·1 and 0·2 for this case.

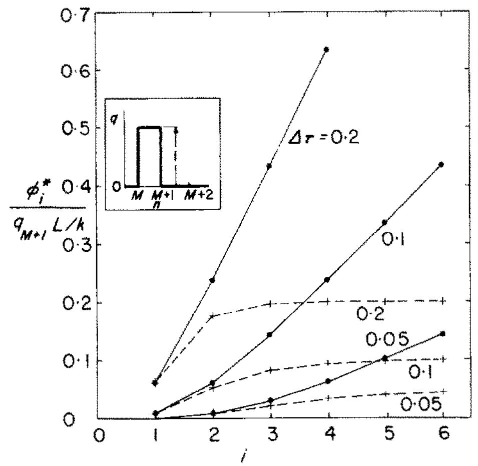

FIG. 5. Dimensionless sensitivity coefficients as a function of the index i for various values of $\Delta\tau$. Dots are for step change in q and crosses are for q shown in inset.

Some insight into the validity of the temporary assumption of constant q given by (7) can also be obtained from Fig. 5. Suppose rather than (7) one lets temporarily

$$q_{M+2} = q_{M+3} = \ldots = q_{M+r} = 0. \tag{14}$$

The analysis for q_{M+1} would proceed exactly in the same as above to obtain (9a) and (13). The only difference would be in the value of the sensitivity coefficient, ϕ_i^*. For this assumption ϕ_i^* is proportional to the partial derivative of T with respect to a unit q of the shape and duration shown in the inset of Fig. 5. The ϕ_i^*'s for this case are shown by the crosses in Fig. 5. Note that in each case, ϕ_1^* is identical in magnitude for both assumptions, (7) or (14). For other ϕ_i^*'s the values for a given $\Delta\tau$ tend to be close together for small i's coupled with small values of $\Delta\tau$. Now the numerical procedure culminating in (9a) and (13) is developed especially for small $\Delta\tau$'s and is intended for use with relatively small values of i or more appropriately r. In Table 1 under the constant q column, r values are given which are recommended on the basis that the assumptions given by (7) or (14) yield about the same ϕ_i^* values and thus q_n's.

Table 1. Recommended r values for constant and linear q assumptions for measurements at the insulated surface of a plane wall

$\Delta\tau$	r Const. q	r Linear q
0·05	3	4
0·1	2	4
0·2	1	3
0·4	0	

Linear q

The above analysis for constant q over Δt can be modified to a linear q assumption. Let q temporarily for $\theta_M < t < \theta_{M+r}$ be given by

$$q = q_M \frac{\theta_{M+1} - t}{\theta_{M+1} - \theta_M} + q_{M+1} \frac{t - \theta_M}{\theta_{M+1} - \theta_M}. \tag{15}$$

This assumption is analogous to that given by equation (7). Note that $q = q_M$ at $t = \theta_M$ and $q = q_{M+1}$ at $t = \theta_{M+1}$. In this case one assumes that q_M is known and q_{M+1} is to be found. With only a difference in the values of the ϕ_i's, one can derive an identical expression to (9a) for this case.

For the q-condition given by (15) one can again derive (11) for the restriction given immediately below the latter equation; the only difference is that now $G(x, t_{\eta+i} - t_\eta)$ represents the temperature rise for a q which is zero at t_η and then increases linearly with time and reaches unity at $t_{\eta+m}$. Figure 6 shows some sensitivity coefficients for this case. A comparison of these values with those in Fig. 5 which are for the constant q assumption shows that the "linear" ϕ_i's are smaller for several time steps; in fact for $\Delta\tau = 0{\cdot}05$ the "constant" ϕ_i^*'s are larger until $i = 4$. This means that the number of "future" temperatures should always be greater for linear q than the constant q assumption. In

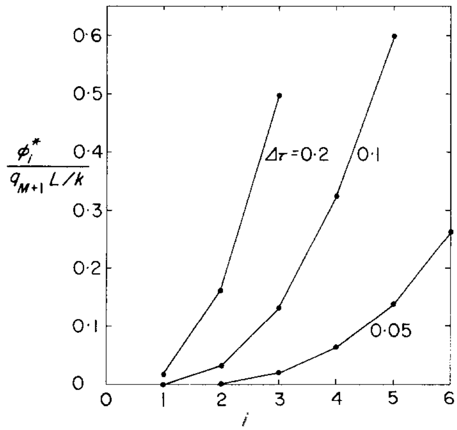

FIG. 6. Sensitivity coefficients for case of linear variation of q with time.

Table 1 are given some r-values that the linear q case must reach in order to equal or exceed the ϕ_i^* values for the constant q assumption. If the same r-values are used for both approximations one finds that the linear q results tend to oscillate more than the constant q results. Depending upon the accuracy of the data, it might be necessary to use larger r values than given in Table 1. This is required when the data is rather inaccurate and is not "smooth". This is discussed further in connection with the examples. Also, it is sometimes necessary to use $r > 1$ when the thermocouple is located at the heated surface ($\alpha t/x^2 \to \infty$) in order to reduce the oscillations in the values of q calculated. Then the least squares procedure provides a linear filter (when the problem is linear) [13].

NONLINEAR ESTIMATION PROCEDURE FOR SIMULTANEOUS DETERMINATION OF $q(t)$

The procedure for the simultaneous determination of all the q_n values describing $q(t)$ is similar to that for one q_n. One minimizes

$$F(\bar{q}) = \sum_{i=1}^{I} (T_i - Y_i)^2 \qquad (16a)$$

where I now is

$$I = mN \qquad (16b)$$

and N is the number of q values and m is defined by (5a). T_i is approximated analogous to (8a) and the method can be developed directly. Since the resulting expression is not recommended, the details are not given. A close examination of this method indicates that the computer time usually will be much greater than the recommended method and that severe stability problems might be encountered. Further discussion of this method is available in an expanded version of this paper available from the author.

EXAMPLES

In the two examples the Crank–Nicolson finite-difference approximation is used since it is quite accurate. For results with other finite-difference approximations see [7]. Also, in both examples, the time step for the data (Δt) and for the heat flux density ($\Delta\theta$) are equal, i.e. $m = 1$ in equation (5). In both cases the properties were assumed to be constant and hence the problems are linear; for these cases similar results would be otained for temperature-variable thermal properties although the computer time would increase. The computation time depends upon the number of future time steps ($r - 1$, for $m = 1$); the computer time will be approximately $2r + 1$ times that for a known boundary condition. The time for most of these cases is about 10–20 s of CDC 3600 time.

"Exact" data for triangular heat flux

The same case of heat flux triangular in shape considered in [7] will be considered here. Unfortunately, the results in [7] show a small lag due to a programming error.

A plate of thickness L is heated at $x = 0$ and insulated at $x = L$. The exact temperatures at these locations are shown in Fig. 7. The heat flux is linear from $\tau = \alpha t/L^2 = 0$ to 0·6,

$$q(t) = q_a \tau$$

where q_a is a constant; q is shown as the solid line in Fig. 8.

A number of calculations have been performed for this case using as the input the temperature history at the insulated surface. In order

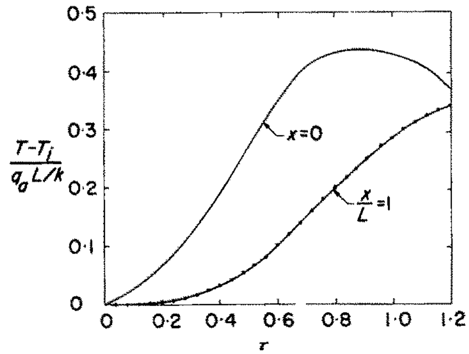

FIG. 7. Temperatures at heated surface and insulated surface for finite plate heated by triangular heat flux.

to simulate an experimental case with measurement errors, the temperatures are *not* introduced as exact numbers; instead the temperatures are truncated at three decimal places with the maximum temperature being 0·34 as shown in Fig. 7. The errors in the temperatures then vary from $-0·001$ to 0 or $-0·3$ to 0 per cent of the maximum temperature rise. The temperatures are introduced at $\Delta\tau = \alpha\Delta t/L^2 = 0·04$ intervals, and the first five T's corresponding to 0, 0·04, 0·08, ... are 0·0, 0·0, 0·0, 0·0 and 0·001. The exact temperature at $\tau = 0·12$ is 0·000374 whereas 0·0 is used; thus the error in the temperature at that time is -100 per cent of the true temperature. Hence, for the early times the errors in the temperature data seem to be relatively large.

For this example the results were not sensitive to the number of spatial nodes greater than 20 or calculational time steps smaller than 0·04. The number of future temperatures was very important with no future temperatures being unstable ($r = 1$), one future temperature ($r = 2$) quite oscillatory (as shown by Fig. 8), and two and three future temperatures ($r = 3, 4$) quite satisfactory as shown in Fig. 9. It appears that three

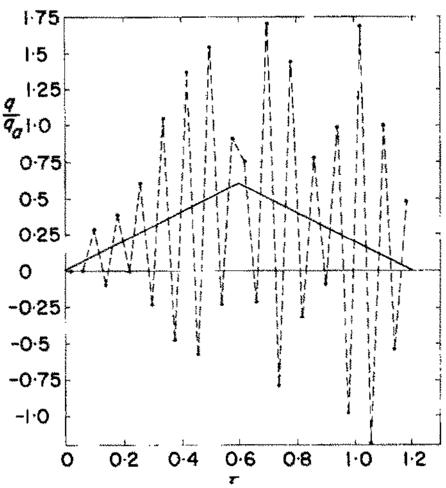

FIG. 8. Calculated heat flux for data used at $x/L = 1$ with $\Delta\tau = 0·04$. Calculation performed with $\Delta\tau = 0·02$ and one future temperature ($r = 2$) used.

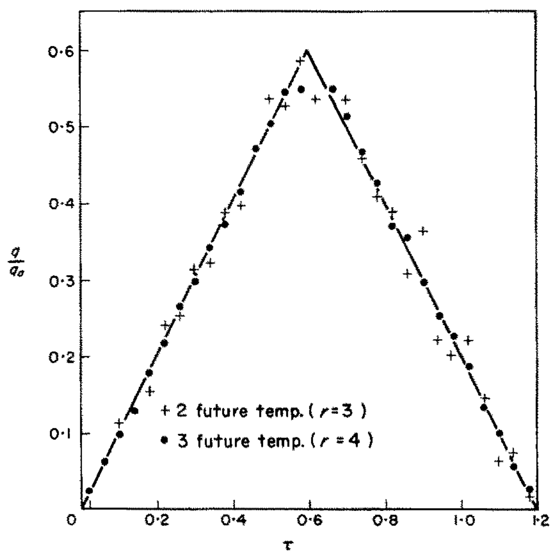

FIG. 9. Calculated heat flux using two and three future temperatures at $x/L = 1$ with data given at $\Delta\tau = 0·04$.

future temperatures are better than two. It is rather remarkable that the heat flux for the first interval (plotted at $\Delta\tau/2$) is as accurate as it is using the first three future temperatures 0·0, 0·0 and 0·001 which are quite inaccurate.

The heated surface temperature is less oscillatory and more accurate than the surface heat flux. This is illustrated by Fig. 10 showing the surface temperature calculated with one future temperature ($r = 2$) and Fig. 8 showing the associated $q(t)$.

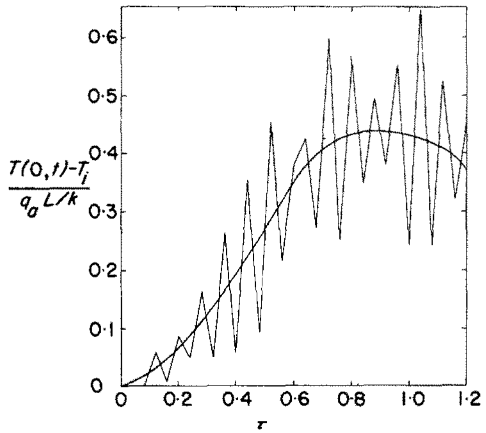

FIG. 10. Calculated surface temperature ($x = 0$) using one future temperature at $x/L = 1$ with data given at $\Delta\tau = 0{\cdot}04$.

Experimental case

An experiment has been run with a copper block initially at 439°K and an aluminium block at 297°K. The blocks were suddenly brought into intimate contact by a hydraulic system. A thin film of water was placed initially on the aluminium surface to improve the contact between the blocks.

The measured temperatures are shown in Fig. 11. Thermocouples are located at 0·00317, 0·00635 and 0·0317 meters from the heated surface of the aluminium block and 0·00317 meter from the surface in the copper block. Measurements were made at 0·2 s intervals. In the aluminium block the dimensionless numbers $\alpha\Delta t/E^2$, where $\Delta t = 0{\cdot}2$ second and E is 0·00317, 0·00635 and 0·0317 meters, are equal to about 1·0, 0·25 and 0·01. Because 0·01 is so small, measurements were used at $\Delta t = 0{\cdot}6$ and are so

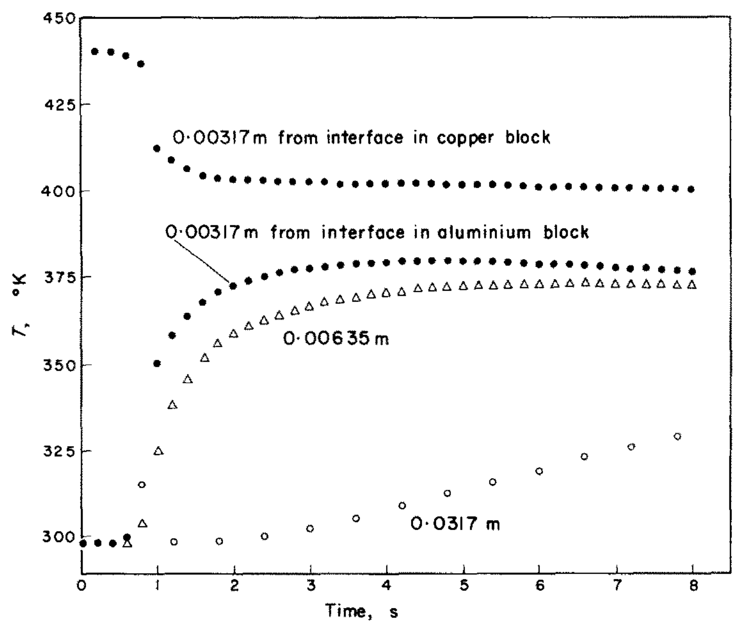

FIG. 11. Experimental temperature data for copper and aluminium blocks in contact.

shown in Fig. 11; this gave $\alpha\Delta t/E^2 \approx 0{\cdot}03$ which is still a little small.

It should be stated that the analytical validity of this method should not be judged mainly on the results of this example because these depend upon the experimental techniques, values of thermal properties, etc. On the other hand a satisfactory technique must provide the capability of treating data of reasonable accuracy.

Depicted in Fig. 12 are the q's calculated for the aluminium block. The agreement between the results from the three thermocouples is very good. The 0·00317 m thermocouple ($\alpha\Delta t/E^2 \approx 1$) should not theoretically need any future temperatures. The results for this case are shown as dots in Fig. 12 and the results for one future temperature by crosses. The former case tends to be more "rough" but might give more accurate results near 0·9 s. The 0·00635 m thermocouple ($\alpha\Delta t/E^2 \approx 0{\cdot}25$) with one future temperature is shown by the triangles in Fig. 12 and gives results very close to those for the first thermocouple.

The instant the specimens come together is not known but may be between 0·3 and 0·5 s. If the contact were perfect, the heat flux would go to infinity at the instant of contact and then decrease rapidly. This is approximated to a limited extent.

For a rapid variation of the heat flux density such as near 0·9 s, the time intervals for the temperature measurements should be small compared to the period of the variation. Perhaps the 0·00317 m thermocouple could have had smaller time steps and then yielded more detail. The q-results obtained from the 0·0317 m thermocouple illustrate the impossibility of accurately calculating the surface flux at times when the time steps for q are on the same order as the period of a rapid variation in q. See the curve in Fig. 12 with the open circles which is for $\Delta t = 0{\cdot}6$ and three future temperatures. The time step $\Delta\theta$ for this case cannot be reduced to 0·2 s without using an excessive number of future temperatures.

The heat flux entering the aluminium block should be that leaving the copper block even though there is a resistance to heat flow at the interface. This is verified experimentally by comparing q shown in Fig. 13 (which is for the

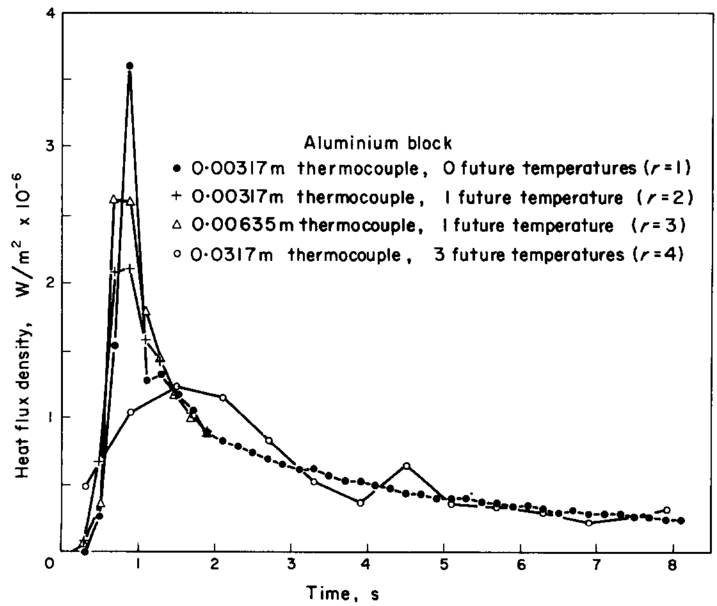

FIG. 12. Calculated heat flux using different thermocouples in the aluminium block.

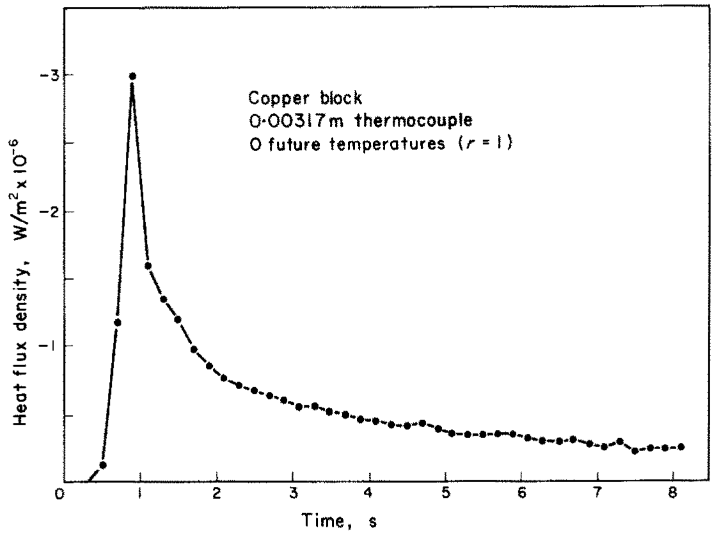

FIG. 13. Calculated heat flux using temperatures in copper block.

copper block) with the q's of Fig. 12. No future temperatures are used in Fig. 13, and none are required from the value of $\alpha\Delta t/E^2 \approx 2$; some of the roughness of the q-curve would be removed with one future temperature, however.

SUMMARY AND CONCLUSIONS

A new numerical different method is given for the nonlinear inverse heat conduction problem. It utilizes the concepts of non-linear estimation. This method may be considered an on-line method since the components of the heat flux are estimated one at a time. A second method is suggested for which all the components of q are found simultaneously.

The sensitivity coefficients can be used to gain insight into these methods. In particular, the sentivity coefficients help indicate that the method for determining one q at a time requires "future" temperatures and further give some insight into the number of future temperatures required.

Examination of the two methods indicates that the single-q method is generally superior. This is significant because several researchers have suggested using some modification of determining the complete q-curve at one time.

In contrast with many previous solutions of the inverse problem, the present analysis clearly demonstrates the capability to treat experimental data. This is demonstrated by two examples, one for which the approximate data was generated analytically and the other which uses experimental data.

In summary this work is unique in several respects. The analysis is for the general non-linear case; the capability of using small dimensionless time steps is demonstrated; and these features are coupled with the capability of treating non-exact data.

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ESTIMATION NON-LINÉAIRE APPLIQUÉE AU PROBLÈME INVERSE DE LA CONDUCTION NON-LINÉAIRE DE LA CHALEUR

Résumé—L'estimation de la température ou de la densité de flux de chaleur de surface, utilisant une histoire de la température mesurée, à l'intérieur d'un solide conducteur de la chaleur, est appelé le problème inverse de la conduction de la chaleur. Ce problème devient non-linéaire si les propriétés thermiques dépendent de la température. On donne une nouvelle méthode de différences finies. Elle est basée en partie sur les concepts d'une technique générale de résolution des problèmes inverses appelée estimation non-linéaire.

La méthode (ou famille de méthodes) estime les composantes du flux de chaleur, une à la fois, et ainsi, peut être considérée comme une méthode successive. Une autre méthode est esquissée pour laquelle toutes les composantes du flux de chaleur sont trouvées simultanément.

Comme il est suggéré par les développements dans l'estimation non linéaire, les coefficients de sensitivité peuvent être utilisés pour comprendre ces méthodes. Les coefficients de sensitivité aident à indiquer que la méthode successive a besoin des températures futures lorsque des pas de temps faibles doivent être employés.

Plusieurs exemples de l'emploi de la méthode successive sont donnés dans les cas où les données ne sont pas exactes. Les résultats montrent que la méthode qui est plutôt remarquable dans sa capacité d'extraire l'information sur la condition de surface à partir de mesures expérimentales qui sont en retard et amorties par rapport à la condition de surface.

NICHTLINEARE ABSCHÄZTUNG FÜR DAS NICHTLINEARE, INVERSE WÄRMELEITUNGSPROBLEM

Zusammenfassung—Die Bestimmung der Oberflächentemperatur oder der Wärmestromdichte aufgrund der Messung einer Temperaturverteilung in wärmeleitenden Körpern wird inverses Wärmeleitungsproblem genannt. Dieses Problem wird nichtlinear, wenn die thermischen Stoffgrössen temperaturabhängig sind. Ein neues Differenzenverfahren wird vorgestellt. Es basiert teilweise auf dem Konzept einer allgemeinen Technik für die Lösung inverser Probleme, auch nichtlineare Abschätzung genannt.

Diese Methode (oder Varianten dieser Methode), bei der die Komponenten des Wärmestroms nacheinander geschätzt werden, kann als Folgeschritt-Verfahren betrachtet werden. Eine andere Methode, bei der alle Komponenten des Wärmestroms gleichzeitig gefunden werden, ist skizziert.

Wie die Entwicklung der nichtlinearen Abschätzung zeigt, wird die Methode bei Verwendung der Einflusskoeffizienten durchsichtiger. Die Einflusskoeffizienten zeigen, dass beim Folgeschritt-Verfahren "zukünftige" Temperaturen benötigt werden, wenn dleine Zeitschritte gewählt werden. An mehreren Beispielen wird die Anwendung des Folgeschrittverfahrens erläutert. Diese Beispiele gelten für Fälle mit nicht genauen Daten. Die Ergebnisse zeigen, dass die vorgeführte Methode bemerkenswerte Informationen über die Oberflächenverhältnisse aus Versuchsmessungen ziehen kann, die gegenüber den Oberflächenwerten phasenverschoben und gedämpft sind.

JAMES V. BECK

НЕЛИНЕЙНАЯ ОЦЕНКА НЕЛИНЕЙНОЙ ОБРАТНОЙ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ

Аннотация—Благодаря измеренной температурной характеристике в теплопроводящем твердом теле обратная задача теплопроводности сводится к определению температуры поверхности или плотности теплового потока. Эта задача становится нелинейной, если термические свойства не зависят от температуры. Приводится новый метод конечных разностей, который частично основан на понятиях общего метода решения обратных задач теплопроводности. Этот метод позволяет оценить изменение компонентов теплового потока во вьемени.

Используая другой метод, можно одновременно найти компоненты теплового потока.

Исходя из нелинейной оценки, коэффициенты чувствительности можно использовать для выявления сущности этих методов.

Коэффициенты чувствительности показывают, что метод последовательных приближений требует сведений о развитии во времени температурного поля, если необходимо использовать небольшие температурные изменения.

Приводятся некоторые из примеров применения метода, которые соответствуют случаям с приближенными данными. Результаты выявили, что метод целесообразно использовать, когда необходимо получить информацию о состоянии поверхности по результатам экспериментов, а также для сравнения и оценки точности условий на поверхности.